Math Review for the M.A. in Economics

Nicholas R. Jenkins
California State University, Long Beach

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Contents

1 Functions

- **Monomials:** (Sometimes called power functions) A function

  $$f(x) = ax^k$$

  where a is a coefficient and k is the degree. Examples: $y = x^2$, $y = \frac{1}{6}z^2$

- **Polynomials:** A sum of monomials.

  $$f(x) = ax^k + bx^k$$

  Examples: $y = x^3 + 4x^2 + \frac{1}{2}x + 5$, $U = 0.5y + 3.8x$

- **Rational Functions:** Ratio of polynomials.
  Examples: $y = \frac{4x}{3}$, $y = \frac{x^3+4x}{x^4-x^3+3}$

- **Exponential Functions:** Examples: $f(x) = 4^n$, $y = e^x$

  Properties of exponential functions:

  1. $a^x a^y = a^{x+y}$
  2. $a^{-x} = \frac{1}{a^x}$
  3. $a^{x-y} = \frac{a^x}{a^y}$
  4. $(a^x)^y = a^{xy}$
  5. $(ab)^x = a^x b^x$
  6. $a^0 = 1$

- **Logarithmic Functions:**

  $$y = \log_a(x)$$

  Examples: $\log_a(a^x) = x$ and $a^{\log_a(x)} = x$

  The common base of the log function is base 10:

  $$y = \log_1 0(x) \iff 10^y = x$$

  Base e is another common base, also known as the natural logarithm:

  $$y = \log_e(x) \iff e^y = x \iff ln(x)$$

Change of Base Formula: Used to switch bases of logs.

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

Properties of logarithmic functions:

1. $\log(xy) = \log(x) + \log(y)$
2. $\log(\frac{x}{y}) = \log(x) - \log(y)$
3. $\log(xy) = \log(x) + log(y)$
4. $\log(x^n) = n\log(x)$
5. $\log(\frac{1}{x}) = \log(x^{-1}) = -\log(x)$
6. $\log(1) = \log(e^0) = 0$
7. $y = \log_b x \Leftrightarrow b^y = x$

- **Trigonometric Functions:** Examples: $y = \sin(x)$, $y = \cos(t)$, $y = \tan(\theta)$

- **Linear Functions:** Polynomial of degree 1.
  Examples: $y = b + mx$, $E(y) = 2.78 + 0.56X_1 + 3.48X_2 + \epsilon$

- **Nonlinear:** Any polynomial of degree greater than 1.
  Examples: $y = x^2 + 4x + \frac{1}{2}$, $y = \cos(x)$, $U(x, y) = 20x^{0.6}y^{0.4}$

- **Summation:**

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n$$

Properties:

1. $\sum_{i=1}^{n} cx_i = c\sum_{i=1}^{n} x_i$
2. $\sum_{i=1}^{n}(x_i + y_i) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i$

There is also a product function: $\prod_{i=1}^{n} x_i = x_1x_2x_3\cdots x_n$ however, we can rewrite this using the summation function if we transform it with logs:

$$\begin{aligned} \log\left(\prod_{i=1}^{n} x_i\right) &= \log(x_1 \cdot x_2 \cdot x_3 \cdots x_n) \\ &= \log(x_1) + \log(x_2) + \log(x_3) + \cdots + \log(x_n) \\ &= \sum_{i=1}^{n} \log(x_i) \end{aligned}$$

This is especially useful with working with likelihood functions used in maximum likelihood estimation. The likelihood function is given as

$$\mathcal{L}(\theta; x_1, \cdots, x_n)$$

which is equal to the joint density functions, given as:

$$P(x_1, x_2, \cdots, |\theta)$$

This gives us:

$$\mathcal{L}(\theta; x_1, \cdots, x_n) = P(x_1, x_2, \cdots, |\theta) = \prod_{i=1}^{n} P(x_i|\theta)$$

Rather than work with the product notation, which is more difficult, we can transform the function with logarithms to get the log-likelihood function, given as:

$$\log\left(\prod_{i=1}^{n} P(x_i|\theta)\right) = \sum_{i=1}^{n} \ln P(x_i|\theta)$$

2 Solving Equations and Finding Inverses

2.1 Linear Equations with One Unknown

Follow the general steps below to solve for linear equations with one unknown. These steps can also be applied to solve equations with more than one unknown:

1. Eliminate any denominators.
2. Eliminate any parentheses.
3. Move the terms which contain the unknown to one side (usually the left side) of the equation, while moving other terms to the opposite side.
4. Collect the terms which contain the same unknown.
5. Make the coefficient of the unknown equal to one.

Example 1: Solve for x.

$$\begin{aligned} \frac{4x + 2(7 - x)}{3} &= 21 \\ 4x + 2(7 - 3) &= 21 \cdot 3 \\ 4x + 14 - 2x &= 63 \\ 4x - 2x &= 63 - 14 \\ 2x &= 49 \\ x &= 24.5 \end{aligned}$$

Note: *Whatever you do to an equation, do the **same** thing to **both** sides of the equation.*

2.2 Polynomial Equation with One Unknown

The basic idea for solving polynomial equation is to reduce or increase the power of the unknown to one. This is accomplished by reducing the power directly, using the quadratic formula, or defining a new unknown.

(1) Reduce or increase the power directly:

Example 2: Solve for x. Both a and c are constants in the equation and $a \neq 0$

$$\begin{aligned} ax^2 - c &= 0 \\ x^2 &= \frac{c}{a} \\ x &= \pm\sqrt{\frac{c}{a}} \end{aligned}$$

Example 3: Solve for x. Both a snd c are constants in the equation and $a \neq 0$.

$$\begin{aligned} ax^3 + c &= 0 \\ x^3 &= -\frac{c}{a} \\ x &= \left(-\frac{c}{a}\right)^{\frac{1}{3}} \end{aligned}$$

(2) Use the quadratic formula: This approach requires getting the equation in the below format, where a, b, and c are constants and $a \neq 0$.

$$ax^2 + bx + c = 0$$

When $b^2 - 4ac \geq 0$, we could solve for x using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example 4:

$$2x^2 + 7x - 4 = 0$$

Since $b^2 - 4ac = 7^2 - 4 \cdot (2) \cdot (-4) = 81 > 0$, solving for x:

$$x = \frac{-7 \pm \sqrt{81}}{2 \cdot 2}$$

$$x_1 = \frac{1}{2}, x_2 = -4$$

Finally, use the conditions from the question to decide which x is the solution or if both of them are solutions.

(3) Define a new unknown: When we cannot reduce or increase the power of the unknown directly, we could define a new unknown, $u = F(x)$.

Example 5: Solve for x.

$$x + x^{\frac{1}{2}} = 6$$

Define $u = x^{\frac{1}{2}}$, then rewrite the equation as:

$$\begin{aligned} u^2 + u &= 6 \\ u^2 + u - 6 &= 0 \\ u &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-6)}}{2 \cdot 1} \\ &= \frac{-1 \pm 5}{2} \end{aligned}$$

$$u_1 = 2, u_2 = -3$$

$$\text{Substitute } x^{\frac{1}{2}} \text{ in for } u \text{ to get } x^{\frac{1}{2}} = 2:$$

$$x^{\frac{1}{2}} = 2(x^{\frac{1}{2}} \geq 0, -3 \text{ is not the solution for u})$$

$$x = 4$$

2.3 Equations with More Than One Unknown

The most common way to solve this type of equation is to use substitution following the steps below.

1. Rewrite the equation in the form $x_i = F(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ so that x_i is a function of other unknowns.
2. Substitute x_i into other equations so that we could substitute out x_i.
3. Repeat substitutions as in (1) and (2) until we have only one unknown.
4. Solve the unknown and substitute back to get other unknowns.

Example 6: Solve for x and y given the two equations (1) and (2).

$$x + y = 8 \quad (1)$$
$$2x + 3y = 17 \quad (2)$$

Solve (1) for x to get:

$$x = 8 - y \quad (3)$$

Substitute (3) into (2)

$$2(8 - y) + 3y = 17 \quad (1)$$
$$16 - 2y + 3y = 17 \quad (2)$$
$$y = 1 \quad (3)$$

Substitute y back into (3):

$$x = 8 - 1, = 7$$

Note: When we rewrite the equation to get $x_i = F(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$, it is easier if we choose the unknown with a simple coefficient.

For some special forms of equations, it is hard to rewrite the unknown into:

$$x_i = F(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n$$

An alternative way is to use one equation to divide, multiply, subtract, or add another equation to easily derive the relationship of two unknowns.

Example 7: Suppose you derive the following equations from the first order conditions of a maximization problem, which has the Cobb-Douglas production function as its objective function.

$$\begin{aligned} \frac{1}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}} &= 6\lambda \quad (1) \\ \frac{2}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}} &= 4\lambda \quad (2) \\ 6K + 4L &= 108 \quad (3) \end{aligned}$$

Divide (1) by (2):

$$\begin{aligned} \frac{\frac{1}{3}K^{-\frac{2}{3}}L^{\frac{2}{3}}}{\frac{2}{3}K^{\frac{1}{3}}L^{-\frac{1}{3}}} &= \frac{6}{4} \\ \frac{L}{2K} &= \frac{6}{4} \\ L &= \frac{12K}{4} \\ L &= 3K \end{aligned}$$

Substitute this relationship into (3):

$$\begin{aligned} 6K + 4(3K) &= 108 \\ 18K &= 108 \\ K &= 6 \end{aligned}$$

Then,

$$L = 18$$

Example 8: Suppose you have the following equations from the first order condition of a utility maximization problem:

$$\begin{aligned} 90x + 90y - 2\lambda &= 0 \quad (1) \\ 90x + 180y - 3\lambda &= 0 \quad (2) \\ 60 - 2x - 3y &= 0 \quad (3) \end{aligned}$$

Notice that the coefficients of x in (1) and (2) are the same. Use (1) - (2) to cancel out $90x$:

$$\begin{aligned} 90x + 90y - 2\lambda - (90x + 180y - 3\lambda) &= 0 \\ -90y + \lambda &= 0 \\ \lambda &= 90y \end{aligned}$$

Substitute λ back into (1):

$$\begin{aligned} 90x + 90y - 2(90y) &= 0 \\ 90x + 90y - 180y &= 0 \\ 90x - 90y &= 0 \\ x &= y \end{aligned}$$

Substitute back into (3):

$$60 - 2x - 3y = 0$$

Since $x = y$, we have:

$$60 - 2x - 3x = 0$$

Since the coefficients of x and y are different in (1) and (2), we need to transform the equations:
Multiply (1) by 2 on both sides to get:

$$2x + 2y = 16$$

Use (2) to minus (1):

$$2x + 3y - (2x + 2y) = 17 - 16$$

Substitute back into (1), thus, $x = 7$.

Practice Problems:

1. $2(x - 3) = 3(x + 2) + 6$, Solve for x.
2. $1 - \frac{x-2}{21} = \frac{2x}{7} - 7$, Solve for x.
3. $\frac{1}{x^2} + 5\left(\frac{1}{x}\right) + 6 = 0$, Solve for x.

   Solve for x, y, and λ

4. $5x + 7 = \lambda$
   $2x + 2y = \lambda$
   $3x + 2y = 15$

2.4 Finding Inverses

In economics, we are often given a demand function $Q_D = 100 - 2P$ and are asked to graph the equation. It is helpful to work with the inverse demand function, $P(Q)$ rather than $Q_D(P)$. To find the inverse, we simply put the equation in terms of P:

$$\begin{aligned} Q_D &= 100 - 2P \\ Q_D + 2P &= 100 \\ 2P &= 100 - Q_D \\ P &= \frac{100 - Q_D}{2} \end{aligned}$$

Practice Problems:

1. Write P in terms of Q. $Q = \frac{1}{2}P + 5$

3 Graphing Linear Equations

To graph a linear function, follow the steps below:

1. Set up the equations according to the conditions given by the questions. Solve for the variable that will be placed on the y-axis, so that the equation has the form $y = mx + b$.

2. Solve for the intercepts on the x and y-axis. The intercept on y-axis is the value of y when x equals to zero. Similarly, the intercept on the x-axis is the value of x when y equals zero. If the equation is in the form $y = mx + b$, the intercept on y-axis is the constant, b.

   This can also be done using the point-slope formula:

   $$y - y_1 = m(x - x_1)$$

3. Determine the slope of the curve. The slope is the change in x over the change in y, or $\frac{\Delta x}{\Delta y}$. Specially, in a linear equation of the form $y = mx + b$, the slope is the coefficient of x, which is m.

4. Draw the curve. Be sure to label the intercepts, slope, the curve itself. Graphing the linear equation, you just need to find the two intercepts and connect the two points.

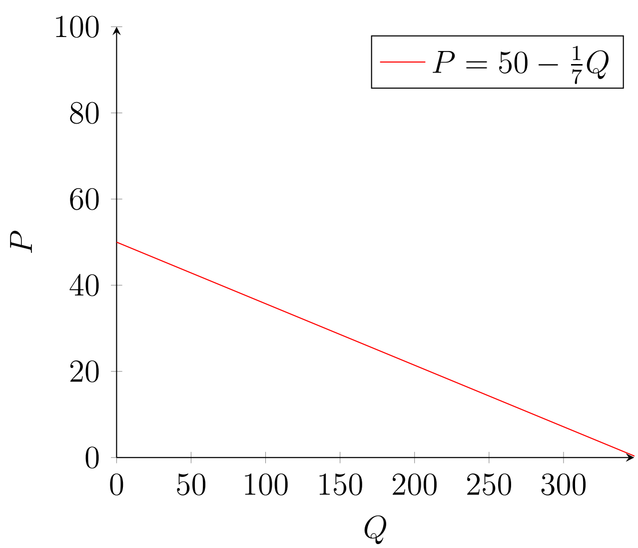
Example 1: Graph the demand curve: $Q = 350 - 7P$
Price will always be on the y-axis and quantity will always be on the x-axis.

1. Solve for P which is on the y-axis: $P = 50 - \frac{1}{7}Q$

2. The slope is $\frac{1}{7} = 0.1429$.

3. The intercept on x-axis is the value of Q when P equals to zero. $Q(0) = 350 - 7(0) = 350$. The intercept on y-axis is the constant in $P = 50 - \frac{1}{7}Q$, which is 50.

4. Graph



Example 2: Graph the total cost Curve: $TC = 15 + 0.5Q$

1. Total costs are in dollars, which will be on the y-axis and the quantity will be on the x-axis. Since the question gives us the equation in terms of $y = mx + b$, we just need to identify the intercepts and the slope.

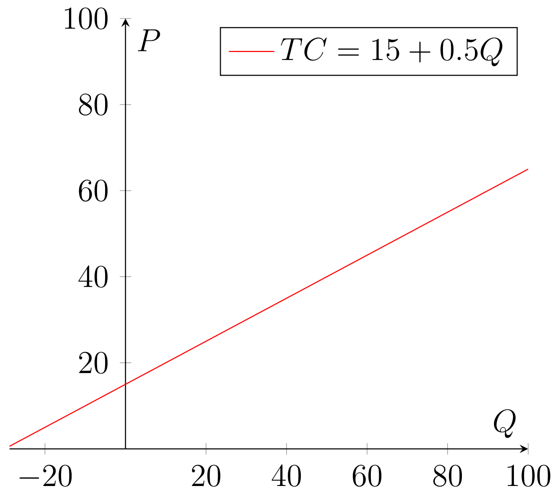
2. We could get the slope directly from the equation, 0.5.

3. The intercept on the x-axis is $TC = 15 + 0.5(0) = 15$.

   We could get the intercept on the x-axis by setting TC equal to 0.

   $0 = 15 + 0.5Q$, we could solve for $Q = -30$.

4. Graph



Note: Negative quantities do not have any meaning in economics. Usually you do not have to show the second quadrant.

Practice Problems:

1. Graph the demand curve: $Q = 120 - 20P$

2. Graph the inverse demand curve: $P = 300 - 0.75Q$

3. Graph the marginal cost curve: $MC = 6Q$.

4. Graph the fixed cost curve: $FC = 7$.

5. Graph the total cost curve: $TC = 100 + 20Q$

4 Differential Calculus

Suppose there are two variables y and x, where the value of y is dependent upon the value of x. The dependence of y upon x means that y is a function of x. This functional relationship is often denoted $y = f(x)$, where f denotes the function. Differentiation is a method to compute the rate at which the variable y changes with respect to the change in the other variable x. This rate of change is called the derivative of y with respect to x. The first derivative can also be interpreted as the slope of the function $f(x)$.

Notation for Derivatives:

$y', f'(x)$	(Prime or Lagrange Notation)
$Dy, Df(x)$	(Operator Notation)
$\frac{dy}{dx}, \frac{df(x)}{dx}$	(Leibniz's Notation)

4.1 Finding the First Derivative

Follow the below rules to find the first derivative of a function:

1. The derivative of a constant is equal to zero.
2. The derivative of a constant times a variable is equal to the constant.
3. For a power function in the form of $y = ax^n$, where a and n are constants, the derivative is:

$$\frac{dy}{dx} = (n)ax^{n-1}$$

   That is, multiply the exponent by the entire function, then reduce the exponent by one.

Example 1: Find $\frac{dy}{dx}$ of $y = 4 + 0.5x^2 + 2x^4$

$$\begin{aligned}\frac{dy}{dx} &= 0 + 2(0.5)x^{2-1} + 4(2)x^{4-1} \\ \frac{dy}{dx} &= x + 8x^3\end{aligned}$$

Properties of derivatives:

1. Power Rule:

$$\frac{dy}{dx}(ax^n) = nax^{n-1}$$

2. Sum Rule:

$$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

3. Constant Rule:
$$\frac{d}{dx}[\alpha f(x)] = \alpha f'(x)$$

4. Product Rule:
$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

5. Quotient Rule:
$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, g(x) \neq 0$$

6. Chain Rule: Let $y = (f \circ g)(x) = [g(x)]$. The derivative of y with respect to x is:
$$\frac{d}{dx}f[g(x)] = f'[g(x)]g'(x)$$

Example 2: Find $\frac{dy}{dx}$ for $y = (3x^2 + 5x - 7)^6$. Let $f(x) = z^6$ and $z = g(x) = 3x^2 + 5x - 7$. Then, $y = f[g(x)]$ and

$$\begin{aligned}\frac{dy}{dx} &= 6(3x^2 + 5x - 7)^{6-1} \cdot (2 \cdot 3x^{2-1} + 5x^{1-1}) \\ &= 6(3x^2 + 5x - 7)^5 \cdot (6x + 5)\end{aligned}$$

Find the first derivative of the following functions:

1. $y = -26 + 0.8x + 4x^2 - 10x^3$
2. $y = -x^2 + 50x^5$
3. $y = 10x^{0.5}$
4. $y = x$
5. $y = 245$

4.2 Derivatives of Exponential and Logarithmic Functions

Recall that the relationship between the logarithmic and the exponential functions is:

$$y = \log_a(x) \Longleftrightarrow a^y = x$$

Also recall the natural logarithm:

$$\ln(x) = \log_e(x)$$

Properties for differentiating exponential functions:

1. $\frac{d}{dx}a^x = a^x \ln(a)$
2. $\frac{d}{dx}\ln(x) = \frac{1}{x}$
3. $\frac{d}{dx}e^x = e^x$

Practice Problems:
Evaluate each of the following functions:

1. $\ln(1)$
2. $\ln(e^2)$
3. $\ln\left(\frac{1}{e}\right)$

Find the first derivative of the following functions:

1. e^{2x+3} *Hint: this requires the chain rule.*
2. $\ln(4x)$
3. $\ln(3x^2 + 5x)$

4.3 Partial Derivatives

Partial derivatives are used when a function has more than one variable. For example, consider a variable y that depends on two variables, x_1 and x_2. This functional relationship is denoted $y = f(x_1, x_2)$. Partial differentiation is a method to compute the rate at which the variable y changes with respect to the one of the other variables, while holding all other variables constant. The partial derivative of y with respect to x_1 is denoted by $\frac{\partial y}{\partial x_1}$. The partial derivative of y with respect to x_2 is denoted by $\frac{\partial y}{\partial x_2}$. The rules for differentiation will stay the same as for functions of one variable. However, say you are taking the partial derivative with respect to x_1, you would treat any other variables (here just x_2) as constants.

<u>Example 3</u>: Find the partial derivatives, $\frac{\partial y}{\partial x_1}$, $\frac{\partial y}{\partial x_2}$, of the function:

$$y = 14 + x_1x_2 + \frac{1}{4}x_2^2$$

For the partial derivative with respect to x_1, the derivative of the constant 14 is zero. Since we should treat x_2 like a constant, the expression x_1x_2 is treated like a constant times a variable, so the partial derivative is just x_2. Finally, the last expression does not contain x_1, so it should be treated as a constant, which has a derivative of zero.

$$\frac{\partial y}{\partial x_1} = 0 + x_2 + 0 = x_2$$

Similarly, for the partial derivative with respect to x_2, the derivative of the constant 14 is zero. Since we should treat x_1 like a constant, the expressionx_1x_2 is treated like a constant times a variable, so the partial derivative is just x_1. Finally, for the last expression, follow the third rule for differentiation.

$$\frac{\partial y}{\partial x_2} = 0 + x_1 + (2)\frac{1}{4}x_2^{2-1} = x_1 + \frac{1}{2}x_2$$

Practice Problems:
Find the partial derivatives, $\frac{\partial y}{\partial x_1} and \frac{\partial y}{\partial x_2}$, of each function.

1. $y = 286 + 7x_1x_2 + x_1^2$
2. $y = x_1 + x_2$
3. $y = 10x_1x_2^3 - 4x_2^5 + 36$
4. $y = 20\sqrt{x_1x_2}$
5. $y = \frac{x_1}{x_2}$

4.4 Total Differentials

The total differential is given by

$$\begin{aligned} df &= \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \cdots + \frac{\partial f}{\partial x_n}dx_n \\ &= f_1dx_1 + f_2dx_2 + \cdots + f_ndx_n. \end{aligned}$$

The total differential states that the total change in some function y is given by the sum of all of the changes in each x value.

Example 1: Find the total differential

$$\begin{aligned} z &= 2x^2y^3 \\ \frac{\partial z}{\partial x} &= 4xy^3 \\ \frac{\partial z}{\partial y} &= 6x^2y^2 \\ so, dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy &= 4xy^3dx + 6x^2y^2dy \end{aligned}$$

Example 2: Find the total differential

$$\begin{aligned} \frac{y^2}{100} &= 100 - \frac{x^2}{4} \\ \frac{2y}{100}dy &= \frac{-2x}{4}dx \end{aligned}$$

Practice Problems: Find the total differential of the function.

1. $t = 4x^6 + 5xy^2$

4.5 Applications of the Derivative: Maxima and Minima

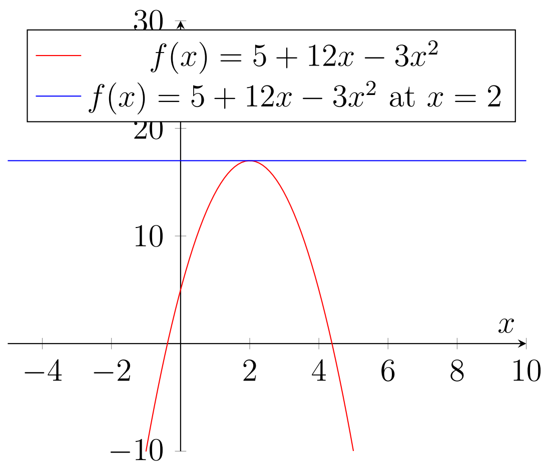
In this section, you will learn how to find the maximum or minimum of a function. The first derivative, $f'(x)$, identifies whether the function $f(x)$ at the point x is increasing or decreasing at x.

Increasing:	$f'(x) > 0$
Decreasing:	$f'(x) < 0$
Neither increasing nor decreasing:	$f'(x) = 0$

The second derivative $f'(x)$ identifies the concavity of the function $f(x)$ at the point x.

Concave down:	$f''(x) < 0$
Concave up:	$f''(x) > 0$

Example 1: Find the value of x that maximizes the function: $f(x) = 5 + 12x - 3x^2$
Graphing the function confirms that the function is a downward facing (concave) parabola.



We need to solve for the point where the tangent line corresponding to $f(x) = 5 + 12x - 3x^2$ has a slope of zero. This will occur at the red point and has a corresponding tangent line represented by the green line.
To find the maximum mathematically, find the first order derivative of the function,

$$f'(x) = 12 - 6x$$

To find the value of x that maximizes the function, set the derivative of the function equal to zero

$$f'(x) = 0$$
$$12 - 6x = 0$$

The solution is obtained by solving the equation for x. Thus,

$$x^* = 2$$

Note that $x^* = 2$ is a called a *critical point.*

Notice that at the maximum, $f\prime(2) = 0$ and $f(x) = 17$.

Since we are assuming that the function is concave, the point $(2, 17)$ is a global maximum. We can verify that this function is indeed concave down by taking the second derivative

$$f''(x) = -6$$

Because this is less than zero, that is $f''(x) < 0$, the function must be concave down.

Example 2: Profit Maximization
Suppose that a plant faces an inverse demand function $P = 40 - 2Q$. The output from the plant is produced at a total cost of $TC = 4Q$.
To determine the level of output (Q) that maximizes this firm's profits we must first recall that profits (π) equal total revenue (TR) minus total cost (TC), where revenue is equal to the inverse demand $(P(Q))$ times output (Q), or

$$\pi(Q) = TR - TC = P(Q)Q - TC$$

.
First, derive the total revenue function

$$\begin{aligned} TR = P(Q)Q &= (40 - 2Q)Q \\ &= 40Q - 2Q^2 \end{aligned}$$

Then, obtaining the profit function

$$\begin{aligned} \pi(Q) &= TR - TC \\ &= 40Q - 2Q^2 - 4Q \\ &= 36Q - 2Q^2 \end{aligned}$$

Now, solve the optimization problem with the techniques developed previously. First, find the first order derivative of the function

$$\pi'(Q) = 36 - 4Q$$

To find the value of output (Q) that maximizes profits, set the first order derivative of our function equal to zero to get the equation

$$\begin{aligned} \pi'(Q) &= 0 \\ 36 - 4Q &= 0 \\ Q^* &= 9 \end{aligned}$$

Notice that at the maximum, $\pi'(9) = 0$ and $\pi(4.5) = 162$.
Lets make sure that we actually found a maximum using the second derivative test.

$$\pi''(Q) = -4$$

Because this is less than zero, that is $f''(x) < 0$, the function must be concave down. Thus, we have in fact found a maximum at the point $(9, 162)$.

Practice Problems:
Maximize each of the functions below and check their concavity (if the are maximum or minimum points).

1. $f(x) = 80 + 100x - 5x^2$
2. $TR(Q) = 100Q - \frac{1}{2}Q^2$
3. $\pi(Q) = -18 + 55Q - 5Q^2$
4. $f(x) = (5 + x)(7 - x)$
5. $\pi(Q) = -aQ^2 + bQ + c$, where a, b, and c are all positive constants.
6. Assume that an individual's total labor compensation, C, is a function their education level, E. Compensation is given by the function $C = -E^2 + 78E - 2E$. Find the value of education, E, that maximizes total labor compensation, C. Verify that the compensation function is concave.

5 Constrained Optimization

Related to the notion of finding maximum and minimum points is that of constrained optimization. This is an essential tool in economics and is used throughout both micro and macroeconomics.

5.1 Lagrangian Multipliers

The goal with constrained optimization problems in n variables is to add an additional variable for each constraint to make it an unconstrained optimization problem in $n + k$ variables. We do this using a Lagrangian multiplier.

The Lagrangian multiplier allows us to write the objective function and the constraint in a single function. Using the Lagrangian multiplier method, the maximization, or minimization, problem can be written as
Given a function of two variables:

$$\begin{array}{ll} \max\limits_{x_1,x_2} / \min & f(x_1, x_2) \\ \text{subject to (s.t)} & c(x_1, x_2) \end{array}$$

or

$$\max_{x_1,x_2} / \min f(x_1, x_2) \text{ s.t. } c(x_1, x_2)$$

We can define the Lagrangian function $\mathcal{L}(x_1, x_2, \lambda)$ as

$$\max_{x_1,x_2,\lambda} / \min \mathcal{L}(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda(c(x_1, x_2))$$

Example 1: Maximize the Cobb-Douglas utility function $U = 0.5X^{0.5}Y^{0.5}$ subject to the constraints of income $I = 9$, price of good x $P_x = 1$, and the price of good y $P_y = 1$.

We begin by writing the budget constraint

$$\begin{aligned} I &= P_x X + P_y Y \\ 9 &= 1X + 1Y \end{aligned}$$

Now, we set the budget constraint equal to zero and write the Lagrangian

$$\max_{x_1,x_2,\lambda} \mathcal{L} = 0.5X^{0.5}Y^{0.5} - \lambda(9 - 1X - 1Y)$$

Now we find the first order conditions

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial X} &= 0.25X^{-0.5}Y^{0.5} - \lambda \\ \frac{\partial \mathcal{L}}{\partial Y} &= 0.25X^{0.5}Y^{-0.5} - \lambda \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= 9 - X - Y \end{aligned}$$

Set the first two first order conditions equal to zero and solve
First Condition:

$$\begin{aligned} 0.25X^{-0.5}Y^{0.5} - \lambda &= 0 \\ 0.25X^{-0.5}Y^{0.5} &= \lambda \end{aligned}$$

Second Condition:

$$\begin{aligned} 0.25X^{0.5}Y^{-0.5} - \lambda &= 0 \\ 0.25X^{0.5}Y^{-0.5} &= \lambda \end{aligned}$$

Setting them equal to each other we get

$$\begin{aligned} 0.25X^{-0.5}Y^{0.5} &= 0.25X^{0.5}Y^{-0.5} \\ X^{-0.5}Y^{0.5} &= X^{0.5}Y^{-0.5} \\ X &= Y \end{aligned}$$

Note: The ratio of the the first and second first order conditions is the marginal rate of substitution. This also yields the same result as above.

$$\begin{aligned} \frac{0.25X^{-0.5}Y^{0.5}}{0.25X^{0.5}Y^{-0.5}} &= \frac{\lambda}{\lambda} \\ \frac{Y}{X} &= 1 \\ Y &= X \end{aligned}$$

Now, substitute this result into the third first order condition

$$\begin{aligned} 9 - X - (X) &= 0 \\ 9 - 2X &= 0 \\ 9 &= 2X \\ 4.5 &= X \\ 4.5 &= Y \end{aligned}$$

Finally, we can substitute the values for X and Y into our utility function to find the utility

$$\begin{aligned} U &= 0.5X^{0.5}Y^{0.5} \\ U &= 0.5(4.5)^{0.5}0.5(4.5)^{0.5} \\ U &= 2.25 \end{aligned}$$

Practice Problems:
Maximize the following functions subject to the given constrains.

1. $U = 0.8X^{0.25}Y^{0.75}$ s.t. $I = 10, P_x = 1, P_y = 2$
2. $U = x^{0.5} + y^{0.5}$ s.t. $I = 100, P_x = 2, P_y = 4$

6 Integral Calculus

6.1 Antiderivatives: The Indefinite Integral

Suppose instead of wanting know the derivative of a function, we want to know what function gives us its derivative. This is known as the antiderivative.

Let $f'(x)$ be the derivative of $f(x)$. Then the antiderivative of $f'(x)$, $f'(x)^{-1}$, is $f(x)$. That is

$$f'(x)^{-1} = f(x)$$

Theorem: If F is an antiderivative of f on an interval I, then the most general antiderivative of F on I is

$$F(x) + c$$

where c is an arbitrary constant.

Examples:

$f(x) = 3x$	$F(x) = \frac{3}{2}x^2$
$f(x) = \frac{1}{2}x^2$	$F(x) = \frac{1}{6}x^3$

If F is the antiderivative of f, then F is also called the indefinite integral of f and can be written as $F(x) = \int f(x)\ dx$.

6.2 The Definite Integral

In economics, we typically focus on the definite integral, which is an integral that specifies upper and lower bounds. The definite integral can be written as $\int_a^b f(x)dx$. This is defined as the area under the curve from $x = a$ to $x = b$.

These two components establish the Fundamental Theorem of Calculus.

Part 1: If the function f is bounded on [a, b] and continuous on (a, b). Then the function

$$F(x) = \int_a^x f(t)dt$$

has a derivative at each point in (a, b) and

$$F'(x) = f(x)$$

Part II: If the function f is bounded an [a, b] and continuous on (a, b) and F is a function that is continuous on [a, b] such that $F'(x) = f(x)$ on (a, b), then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Rules of integration:

- $\int x^n dx = \frac{1}{n+1}x^{n+1} + c$
- $\int e^x dx = e^x + c$
- $\int \frac{1}{x} dx = \ln(x) + c$
- $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$

Example 1:

$$\begin{aligned} \int_1^3 3x^2 dx &= \\ &= \frac{1}{2+1} 3x^{2+1} \Big|_1^3 \\ &= x^3 \Big|_1^3 \\ &= (3)^3 - (1)^3 \\ &= 27 - 3 \\ &= 24 \end{aligned}$$

Integrals are useful for finding the area under a function or the area between two functions. Rather than having to divide the area under a curve an infinite number of times, as in the case of Riemann Sums.

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

Properties of definite integrals:

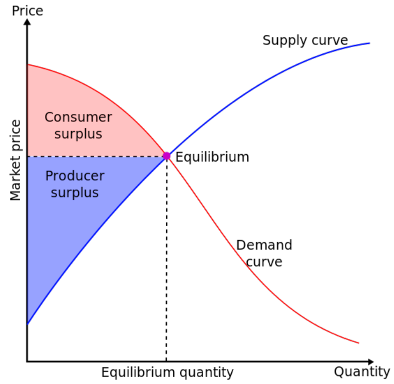
1. Switching the limits of integration reverses the sign: $\int_a^b f(x)dx = -\int_b^a f(x)dx$
2. If c is a constant $\int_a^b cdx = c(b - a)$
3. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
4. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
5. $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

Practice Problems

1. $\int_2^4 4xdx$
2. $\int_0^5 \frac{1}{3}xdx$

6.3 Applications of Integration: Economic Welfare

We can use integration to find the consumer and producer surplus in an economic market.



Suppose the demand curve for the market is given by $20Q + 3$, the suply curve is given by $Q + 1$ and the equilibrium price is 4 and the equilibrium quantity is 1. Integrating across the x-axis, we can formulate our problem as

$$
\begin{aligned}
CS &= \int_a^b D(Q)dQ \\
&= \int_0^1 (20Q + 3)dQ \\
&= \frac{20}{2}Q^2 + 3Q\Big|_0^1 \\
&= \big(10(1)^2 + 3(1)\big) - \big(10(0)^2 + 3(0)\big) \\
CS &= 13
\end{aligned}
$$

To find the producer surplus, we need to subtract the area under the supply curve from the total revenue in the market $P_e * Q_e$. This is given by

$$
\begin{aligned}
PS &= P_e \cdot Q_e - \int_a^b S(Q)dQ \\
&= (4)(1) - \int_0^1 (Q + 1)dQ \\
&= 4 - \left(\frac{1}{2}Q^2 + 1Q\right)\Big|_0^1 \\
&= 4 - \left[\left(\frac{1}{2}(1)^2 + 1(1)\right) - \left(\frac{1}{2}(0)^2 + 1(0)\right)\right] \\
&= 4 - \left[\frac{3}{2} - 0\right] \\
&= 4 - \left[\frac{3}{2}\right] \\
PS &= \frac{5}{2}
\end{aligned}
$$

7 Short Introduction to Linear Algebra

7.1 Vectors

A vector is simply a row or column of data. Technically, it is in n-space as an ordered row or column of numbers. The number of elements in a vector is known as the dimension of the vector.

$$\text{Column vector: } \mathbf{v} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ \frac{1}{2} \end{pmatrix}$$

$$\text{Row vector: } \mathbf{v} = \begin{pmatrix} 1 & 3 & 4 & \frac{1}{2} \end{pmatrix}$$

Vectors are typically denoted with bold face letters on computers, $\mathbf{v}$. However, with pencil and paper, they are denoted with arrows, $\vec{v}$.

7.1.1 Vector Addition and Subtraction

If two vectors $\vec{u}$ and $\vec{v}$ have the same length (the same number of elements), they can be added or subtracted together.

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 & u_2 + v_2 & \cdots & u_n + v_k \end{pmatrix}$$

$$\vec{u} - \vec{v} = \begin{pmatrix} u_1 - v_1 & u_2 - v_2 & \cdots & u_n - v_k \end{pmatrix}$$

7.1.2 The Dot Product

The dot product is given by

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_k v_n = \sum_{i=1}^{n} u_i v_i$$

7.1.3 Scalar Multiplication

A scalar is a vector with a single element $\vec{v} = (5)$. A scalar is also known as a constant. The product of a scalar and a vector is given by

$$c\mathbf{u} = \begin{pmatrix} cu_1 & cu_2 & \cdots & cu_n \end{pmatrix}$$

7.2 Matrices

A matrix is a collection of vectors arranged in m rows and n columns. The dimensionality of a matrix is the number of rows by the number of columns mxn.

$$\mathbf{A} = \begin{pmatrix} a_1 1 & a_1 2 & \cdots & a_1 n \\ a_2 1 & a_2 2 & \cdots & a_2 n \\ \vdots & \vdots & \ddots & \vdots \\ a_m 1 & a_m 2 & \cdots & a_m n \end{pmatrix}$$

7.2.1 Matrix Addition

Suppose we have two matrices **A** and **B**, then

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_1 1 + b_1 1 & a_1 2 + b_1 2 & \cdots & a_1 n + b_1 n \\ a_2 1 + b_2 1 & a_2 2 + b_2 2 & \cdots & a_2 n + b_2 n \\ \vdots & \vdots & \ddots & \vdots \\ a_m 1 + b_m 1 & a_m 2 + b_m 2 & \cdots & a_m n + b_m n \end{pmatrix}$$

7.2.2 Matrix Multiplication

Compute $\mathbf{A} \cdot \mathbf{B}$ if

$\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}$

and

$\mathbf{B} = \begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix}$

$$\begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix} \cdot \begin{pmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{pmatrix} = \begin{pmatrix} (2 \cdot 4) + (3 \cdot 1) & (2 \cdot 3) + (3 \cdot -2) & (2 \cdot 6) + (3 \cdot 3) \\ (1 \cdot 4) + (-5 \cdot 1) & (1 \cdot 3) + (-5 \cdot -2) & (1 \cdot 6) + (-5 + 3) \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{pmatrix}$$